

# LINEAR VERSUS SET VALUED KRONECKER REPRESENTATIONS

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**ABSTRACT.** A set valued representation of the Kronecker quiver is nothing but a quiver. We apply the forgetful functor from vector spaces to sets and compare linear with set valued representations of the Kronecker quiver.

## 1. INTRODUCTION

We consider the Kronecker quiver

$$\mathbf{K}_2 \quad \circ \rightrightarrows \circ$$

and study its representations. Given any category  $\mathbf{C}$ , a Kronecker representation in  $\mathbf{C}$  is by definition a pair of parallel morphisms

$$X_1 \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} X_0$$

in  $\mathbf{C}$ . We may view  $\mathbf{K}_2$  as a category with two objects, two identical morphisms, and two parallel morphisms that connect both objects. Then a Kronecker representation is nothing but a functor  $\mathbf{K}_2 \rightarrow \mathbf{C}$  and the morphisms between representations are by definition the natural transformations. We denote the category of Kronecker representations by  $\mathbf{C}^{\mathbf{K}_2}$ . There are two natural examples.

**Set valued Kronecker representations.** Take for  $\mathbf{C}$  the category **Set** of sets. Then a Kronecker representation is just a quiver, because a quiver is a quadruple  $\Gamma = (\Gamma_0, \Gamma_1, s, t)$  consisting of a set of vertices  $\Gamma_0$ , a set of arrows  $\Gamma_1$  and two maps  $s, t: \Gamma_1 \rightarrow \Gamma_0$  that assign to each arrow its start and its terminus [2]. Thus  $\mathbf{Set}^{\mathbf{K}_2}$  identifies with the category of quivers.

**Linear Kronecker representations.** Fix a field  $k$  and consider for  $\mathbf{C}$  the category  $\mathbf{Vec}_k$  of vector spaces over  $k$ . Then the Kronecker representations are pairs of  $k$ -linear maps, and these have been studied by Leopold Kronecker [3].

**The forgetful functor and its adjoint.** Any functor  $F: \mathbf{C} \rightarrow \mathbf{D}$  induces a functor  $F^{\mathbf{K}_2}: \mathbf{C}^{\mathbf{K}_2} \rightarrow \mathbf{D}^{\mathbf{K}_2}$  by composing representations in  $\mathbf{C}$  with  $F$ .

There is an adjoint pair of functors

$$|-|: \mathbf{Vec}_k \longrightarrow \mathbf{Set} \quad \text{and} \quad k^{(-)}: \mathbf{Set} \longrightarrow \mathbf{Vec}_k.$$

The first one is the forgetful functor and the second one is its left adjoint which takes a set  $X$  to the vector space  $k^{(X)}$  of maps  $f: X \rightarrow k$  such  $f(x) = 0$  for all but a finite number of elements  $x \in X$ . Composing representations with these functors yields another pair of adjoint functors

$$|-|^{\mathbf{K}_2}: \mathbf{Vec}_k^{\mathbf{K}_2} \longrightarrow \mathbf{Set}^{\mathbf{K}_2} \quad \text{and} \quad k^{(-)^{\mathbf{K}_2}}: \mathbf{Set}^{\mathbf{K}_2} \longrightarrow \mathbf{Vec}_k^{\mathbf{K}_2}$$

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but we simplify (and abuse) notation by writing  $|-|$  instead of  $|-|^{K_2}$  and similarly  $k^{(-)}$  instead of  $k^{(-)^{K_2}}$ .

The subject of this work is the study of the forgetful functor and its left adjoint between the categories of linear and set valued Kronecker representations. We restrict ourselves to representations in the category of finite dimensional vector spaces and the category of finite sets.

## 2. LINEAR REPRESENTATIONS

Fix a field  $k$  and let  $\mathbf{vec}_k$  denote the category of finite dimensional vector spaces over  $k$ . For a vector space  $V$  set  $V^* := \text{Hom}_k(V, k)$ .

We consider the category  $\mathbf{vec}_k^{K_2}$  of  $k$ -linear Kronecker representations. From the Krull-Remak-Schmidt theorem it follows that each representation decomposes essentially uniquely into a finite direct sum of indecomposable representations.

It is well known that each indecomposable representation is either *preprojective*, *preinjective*, or *regular*. Following an idea of A. Hubery, we describe these as follows.

For each integer  $n \geq 0$  let  $V_n$  denote the  $n+1$ -dimensional space of homogeneous polynomials of degree  $n$  in two variables  $x$  and  $y$  of degree 1. It is convenient to set  $V_{-1} = 0$ . Thus

$$k[x, y] = \bigoplus_{n \geq 0} V_n.$$

The indecomposable preprojective representations are of the form

$$P(n) \quad V_{n-1} \begin{array}{c} \xrightarrow{x} \\ \xrightarrow{y} \end{array} V_n \quad (n \geq 0)$$

and the indecomposable preinjective representations are the dual representations

$$I(n) \quad V_n^* \begin{array}{c} \xrightarrow{x^*} \\ \xrightarrow{y^*} \end{array} V_{n-1}^* \quad (n \geq 0).$$

Each  $0 \neq f \in V_n$  gives rise to a regular representation

$$R(f) \quad V_{n-1} \begin{array}{c} \xrightarrow{x} \\ \xrightarrow{y} \end{array} V_n / \langle f \rangle$$

where  $\langle f \rangle$  is the  $k$ -linear subspace generated by  $f$ . Note that  $R(fg) \cong R(f) \oplus R(g)$  when  $f$  and  $g$  are coprime. If  $f = p^d$  for some irreducible polynomial  $p$  and an integer  $d \geq 1$ , then  $R(f)$  is indecomposable, and all indecomposable regular representations are of this form.

## 3. THE FORGETFUL FUNCTOR

Let  $\mathbf{set}$  denote the category of finite sets. We fix a finite field  $k$  and consider the forgetful functor  $|-|: \mathbf{vec}_k^{K_2} \rightarrow \mathbf{set}^{K_2}$ . It takes a linear representation

$$X_1 \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} X_0$$

to the set valued representation

$$|X_1| \begin{array}{c} \xrightarrow{|f|} \\ \xrightarrow{|g|} \end{array} |X_0|.$$

This functor preserves products since it is a right adjoint. Thus it suffices to describe its action on the indecomposable linear representations.

Let us introduce two types of quivers that are relevant, namely *linear* and *cyclic* quivers:

$$\begin{array}{lcl} A_r & 1 & \longrightarrow 2 \longrightarrow \cdots \longrightarrow r-1 \longrightarrow r \\ C_r & 1 & \begin{array}{c} \xrightarrow{\quad} 2 \longrightarrow \cdots \longrightarrow r-1 \longrightarrow r \\ \quad \quad \quad \curvearrowright \end{array} \end{array}$$

**Preprojective representations.** Fix an integer  $n \geq 0$  and  $f \in V_n$ . We set

$$d_x(f) := \max\{r \geq 0 \mid x^r \text{ divides } f\} \quad \text{and} \quad d_y(f) := \max\{r \geq 0 \mid y^r \text{ divides } f\}.$$

If  $f = 0$  or  $d_y(f) = 0$ , then we define a quiver  $\Gamma(f)$  as follows. For  $f = 0$  set

$$\Gamma(f) \quad 0 \curvearrowright 0$$

and for  $d_y(f) = 0$  set  $d := d_x(f)$  and

$$\Gamma(f) \quad f \xrightarrow{x^{-1}f} x^{-1}fy \xrightarrow{x^{-2}fy} \cdots \xrightarrow{x^{-d}fy^{d-1}} x^{-d}fy^d.$$

Note that  $\Gamma(f)$  is a linear quiver with  $d_x(f) + 1$  vertices when  $f \neq 0$ .

**Proposition 3.1.** *The quiver  $|P(n)|$  equals the disjoint union of the quivers  $\Gamma(f)$  where  $f \in V_n$  such that  $f = 0$  or  $d_y(f) = 0$ .*

*Proof.* At each vertex there is at most one arrow starting and at most one arrow ending, since multiplication with  $x$  and  $y$  provides injective maps  $V_{n-1} \rightarrow V_n$ . Thus each connected component of  $|P(n)|$  is either linear or cyclic, and it is easily checked that each component is identified by the vertex  $f \in V_n$  satisfying  $f = 0$  or  $d_y(f) = 0$ .  $\square$

Let  $(A, a)$  be a pointed set and  $n \geq 0$ . We define a quiver  $\Gamma((A, a), n)$  with set of vertices  $A^{n+1}$  and set of arrows  $A^n$  such that the start and terminus of an arrow  $\mathbf{a} = (a_1, a_2, \dots, a_n)$  are given as follows:

$$(a, a_1, \dots, a_n) \xrightarrow{\mathbf{a}} (a_1, \dots, a_n, a)$$

**Corollary 3.2.** *View the field  $k$  as pointed set  $(k, 0)$ . Then the quiver  $|P(n)|$  identifies with  $\Gamma((k, 0), n)$ .*  $\square$

**Example 3.3.** Let  $k = \mathbb{F}_2$ . Then  $|P(2)|$  has 3 connected components.

$$\begin{array}{ccc} (0, 0, 0) & \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} & (0, 0) \quad (0, 1, 1) \xrightarrow{(1,1)} (1, 1, 0) \\ (0, 0, 1) & \xrightarrow{(0,1)} (0, 1, 0) & \xrightarrow{(1,0)} (1, 0, 0) \end{array}$$

**Preinjective representations.** Fix an integer  $n \geq 0$ . Following [1], the *de Bruijn graph* of dimension  $n$  on a set  $A$  is the quiver with set of vertices  $A^n$  and set of arrows  $A^{n+1}$  such that the start and terminus of an arrow  $\mathbf{a} = (a_0, a_1, \dots, a_n)$  are given as follows:

$$(a_0, \dots, a_{n-1}) \xrightarrow{\mathbf{a}} (a_1, \dots, a_n)$$

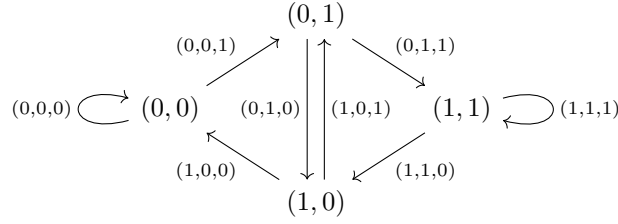
**Proposition 3.4.** *The quiver  $|I(n)|$  identifies with the de Bruijn graph of dimension  $n$  on the set  $k$ .*

*Proof.* The identification goes as follows. We use dual basis elements  $\xi^i \psi^j$  in  $V_n^*$  corresponding to  $x^i y^j$  in  $V_n$ , and we identify

$$k^{n+1} \xrightarrow{\sim} V_n^*, \quad (a_0, a_1, \dots, a_n) \mapsto \sum_{i+j=n} a_j \xi^i v^j.$$

The map  $x^*: V_n^* \rightarrow V_{n-1}^*$  sends  $\sum_{i+j=n} a_j \xi^i v^j$  to  $\sum_{i+j=n} a_j \xi^{i-1} v^j$ , while  $y^*$  sends  $\sum_{i+j=n} a_j \xi^i v^j$  to  $\sum_{i+j=n} a_j \xi^i v^{j-1}$ , where  $\xi^{-1} = 0 = v^{-1}$ .  $\square$

**Example 3.5.** Let  $k = \mathbb{F}_2$ . Then  $|I(2)|$  is the de Bruijn graph of dimension 2 on a set with 2 elements.



**Regular representations.** We fix  $0 \neq f \in V_n$  and consider the corresponding regular representation

$$R(f) \quad V_{n-1} \begin{array}{c} \xrightarrow{x} \\ \xrightarrow{y} \end{array} V_n / \langle f \rangle$$

Suppose that  $R(f)$  is indecomposable. Then one of the maps given by multiplication with  $x$  and  $y$  is bijective. First we consider the case that both maps are bijective.

**Lemma 3.6.** *Let  $\Gamma = (\Gamma_0, \Gamma_1, s, t)$  be a finite quiver. Then the maps  $s$  and  $t$  are bijective if and only if  $\Gamma$  is a disjoint union of cyclic quivers.*

*Proof.* The maps  $s$  and  $t$  are bijective if and only if at each vertex there is precisely one arrow starting and one arrow ending.  $\square$

**Proposition 3.7.** *Let  $R(f)$  be a regular representation and suppose that the maps given by multiplication with  $x$  and  $y$  are bijective. Then the quiver  $|R(f)|$  is a disjoint union of cyclic quivers.*

*Proof.* Apply Lemma 3.6.

**Example 3.8.** Let  $a \in k^\times$  and consider the representation

$$k \begin{array}{c} \xrightarrow{1} \\ \xrightarrow{a} \end{array} k$$

which is isomorphic to  $R(ax - y)$ . For  $k = \mathbb{F}_5$  and  $a = 4$  the corresponding quiver is the following:

$$0 \curvearrowright 0 \qquad 1 \begin{array}{c} \xrightarrow{1} \\ \xleftarrow{4} \end{array} 4 \qquad 2 \begin{array}{c} \xrightarrow{2} \\ \xleftarrow{3} \end{array} 3$$

For  $k = \mathbb{F}_5$  and  $a = 2$  one obtains the following:

$$0 \curvearrowright 0 \qquad 1 \xrightarrow{1} 2 \xrightarrow{2} 4 \xrightarrow{4} 3$$

It remains to consider the case that one of the maps given by multiplication with  $x$  and  $y$  is not bijective. We may assume that this is  $y$ . Note that the kernel is isomorphic to  $k$ . The endomorphism of  $V_n/\langle f \rangle$  given by multiplication with  $x^{-1}y$  is nilpotent, say of index  $d$ .

We consider the quiver  $|R(f)|$  and set  $q := \text{card}(k)$ . Observe that at each vertex there is precisely one arrow starting and there are either  $q$  arrows ending or none.

Given a vertex  $v$  that is not the end of an arrow, there is a unique path of length  $d$  that starts at  $v$  and ends at 0.

A quiver is called a *complete directed tree of height  $d$  and width  $q$* , if

- (1) there is a unique vertex (called *root*) where no arrow starts;
- (2) at each other vertex a unique arrow starts;
- (3) at each vertex either  $q$  or no arrows end;
- (4) each vertex where no arrow ends is connected with the root by a unique path of length  $d$ .

**Proposition 3.9.** *Let  $R(f)$  be an indecomposable regular representation such that  $f = p^d$  for some irreducible polynomial  $p$ . Suppose that the map given by multiplication with  $y$  is not bijective. Then the quiver  $|R(f)|$  has a unique loop at 0, and after removing this loop it is a directed tree that is complete of height  $d$  and width  $\text{card}(k)$ , except that there are only  $\text{card}(k) - 1$  arrows ending at the root 0.*  $\square$

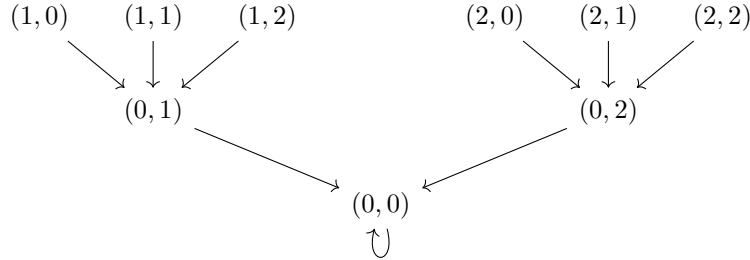
If  $R(f)$  is an indecomposable regular representation such that multiplication with  $x$  is not bijective, then one obtains  $|R(f)|$  from  $|R(f')|$  by reversing all arrows, where  $f'$  is obtained from  $f$  by interchanging  $x$  and  $y$ .

A quiver that is isomorphic or anti-isomorphic to one arising in Proposition 3.9 is called *almost complete directed tree of width  $q$* .

**Example 3.10.** Consider the representation

$$k^2 \begin{array}{c} \xrightarrow{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} \\ \xrightarrow{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}} \end{array} k^2$$

which is isomorphic to  $R(y^2)$ . For  $k = \mathbb{F}_3$  the corresponding quiver is the following:



#### 4. THE FORGETFUL FUNCTOR AND ITS ADJOINT

Let  $k$  be a finite field with  $q$  elements. Composing the forgetful functor and its left adjoint yields for any quiver  $\Gamma$  a natural monomorphism  $\Gamma \rightarrow |k^{(\Gamma)}|$ . Combining this with the description of the quivers arising from indecomposable linear Kronecker representations, we obtain the following result about finite quivers.

**Corollary 4.1.** *Every finite quiver embeds naturally into a product of quivers whose connected components belong to the following list:*

- (1) linear quivers of type  $A_n$  with  $n > 0$ ,
- (2) cyclic quivers of type  $C_n$  with  $n > 0$ ,
- (3) de Bruijn graphs of dimension  $n \geq 0$  on a set with  $q$  elements,
- (4) almost complete directed trees of width  $q$ .

*Proof.* Let  $\Gamma$  be a finite quiver. Then the linear representation  $k^{(\Gamma)}$  decomposes into a finite direct sum of indecomposable representations by the Krull-Remak-Schmidt theorem. It remains to observe that direct sums are products in the category of linear representations and that the forgetful functor preserves products.  $\square$

## 5. LINEARISATION

Fix a field  $k$ . We consider the functors

$$\mathbf{Set} \longrightarrow \mathbf{Vec}_k, \quad X \mapsto k^{(X)}, \quad \text{and} \quad \mathbf{Set}^{\text{op}} \longrightarrow \mathbf{Vec}_k, \quad X \mapsto k^X.$$

These functors induce functors between the categories of set valued and linear Kronecker representations.

**Lemma 5.1.** *For any set  $X$  we have a natural isomorphism  $k^X \cong \text{Hom}_k(k^{(X)}, k)$ .*

*Proof.* Both functors agree on finite sets and preserve colimits. It remains to observe that any set is a coproduct of finite sets.  $\square$

Let us consider the linear representations corresponding to the linear quiver  $\mathbf{A}_n$ .

**Proposition 5.2.** *Let  $n \geq 0$ . Then we have*

$$k^{(\mathbf{A}_{n+1})} \cong P(n) \quad \text{and} \quad k^{\mathbf{A}_{n+1}} \cong I(n).$$

*Proof.* For any  $m \geq 0$  we consider the basis  $\{x^i y^j \mid i + j = m\}$  of  $V_m$ . The linear quiver  $\mathbf{A}_{n+1}$  identifies with

$$\{x^i y^j \mid i + j = n - 1\} \xrightleftharpoons[y]{x} \{x^i y^j \mid i + j = n\}$$

and this yields the isomorphism  $k^{(\mathbf{A}_{n+1})} \xrightarrow{\sim} P(n)$ . The second isomorphism follows from the first with Lemma 5.1.  $\square$

Next consider the representation corresponding to the cyclic quiver  $\mathbf{C}_n$ . It is easily seen that  $k^{(\mathbf{C}_n)} \cong k^{\mathbf{C}_n}$  is isomorphic to the regular representation

$$k^n \xrightleftharpoons[\begin{smallmatrix} 0 & 1 \\ I_{n-1} & 0 \end{smallmatrix}]{I_n} k^n$$

which is isomorphic to  $R(y^n - 1)$ . Let  $p$  denote the characteristic of  $k$  and write  $n = p^c m$  with  $m$  coprime to  $p$ . Then there are pairwise coprime irreducible polynomials  $f_1, \dots, f_t$  such that  $y^n - 1 = (y^m - 1)^{p^c} = f_1^{p^c} \cdots f_t^{p^c}$ . This yields a decomposition of  $R(y^n - 1)$ .

**Proposition 5.3.** *Let  $n > 0$ . Then we have the following decomposition into indecomposable regular representations:*

$$k^{(\mathbf{C}_n)} \cong R(f_1^{p^c}) \oplus \dots \oplus R(f_t^{p^c})$$

*Proof.* Use that  $R(fg) \cong R(f) \oplus R(g)$  when  $f$  and  $g$  are coprime homogeneous polynomials.  $\square$

## 6. PATH ALGEBRAS

Let  $\mathbf{Ring}$  denote the category of associative rings with unit. Given a Kronecker representation

$$A \xrightleftharpoons[g]{f} B$$

in  $\mathbf{Ring}$ , we view  $B$  as a bimodule over  $A$  with left multiplication given by  $f$  and right multiplication given by  $g$ . Taking this representation to the corresponding tensor algebra

$$T_A(B) = \bigoplus_{n \geq 0} B^{\otimes n}$$

yields a functor

$$T: \mathbf{Ring}^{\mathbf{K}_2} \longrightarrow \mathbf{Ring}.$$

The functor

$$\mathbf{Set}^{\mathrm{op}} \times \mathbf{Ring} \longrightarrow \mathbf{Ring}, \quad (X, A) \mapsto A^X$$

induces a functor

$$(\mathbf{Set}^{K_2})^{\mathrm{op}} \times \mathbf{Ring} \longrightarrow \mathbf{Ring}^{K_2}.$$

Given a commutative ring  $A$  and a quiver  $\Gamma$ , the *path algebra*  $A[\Gamma]$  is the free  $A$ -module with basis consisting of all paths in  $\Gamma$  and with multiplication induced by concatenation.

**Proposition 6.1.** *The composition*

$$(\mathbf{Set}^{K_2})^{\mathrm{op}} \times \mathbf{Ring} \longrightarrow \mathbf{Ring}^{K_2} \xrightarrow{T} \mathbf{Ring}$$

*takes a pair  $(\Gamma, A)$  consisting of a finite quiver and a commutative ring to its path algebra  $A[\Gamma]$ .*

*Proof.* Let  $\Gamma = (\Gamma_0, \Gamma_1, s, t)$  be a finite quiver. The path algebra  $A[\Gamma]$  identifies with the tensor algebra  $T_{A\Gamma_0}(B)$  where  $B$  is the free  $A$ -module with basis  $\Gamma_1$ .  $\square$

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